Flow Behavior of Viscoelastic Fluids in the Inlet Region of a Channel

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The behavior of viscoelastic fluids in the inlet region of a channel formed by two parallel flat plates has been analyzed with a Rivlin-Ericksen approximation to describe the fluid properties. It was found that viscoelasticity changed the entry length; both the magnitude and direction of this change were predicted to depend upon the detailed fluid properties. These predictions of the change in the entry length were seen to find some qualitative support in available experimental evidence.

The *directions for further analysis, based primarily on the important observation that* these materials may exhibit solidlike behavior in the immediate vicinity of the entrance, are indicated. This solidlike behavior is seen to be closely related to "melt fracture" and similar instability phenomena common to these materials; it serves to explain directly the otherwise perplexing observations concerning the similarity of such fluid fracture phenomena in materials as divergent in their properties as viscoelastic polymeric systems and inelastic dilatant slurries.

A currently important engineering problem is that of the analysis of the flow behavior of viscoelastic fluids in complex geometries. While a number of problems of interest in the polymer processing industry may be analyzed at least approximately in terms of laminar shearing motions (*12, 16, 42, 58*), most problems of engineering interest are far more complex. Examples of such complex motions are found in the turbulent flow of dilute polymer solutions in a pipe (*28, 46*); the behavior of bulk polymers in extruders, mills, and mixers (*24*); and in fiber spinning operations (*56*). In this paper, we shall investigate one such fluid dynamic problem: the flow of a viscoelastic material in the entrance region of a conduit.

PREVIOUS WORK

The entrance length problem for the flow of *Newtonian* fluids into conduits has attracted the efforts of a number of researchers during the past half century, many of whom have made use of the Prandtl–von Karman boundary-layer theory (*48*) and have applied it in computing the velocity and pressure fields. Schiller (*47*) used the von Karman–Pohlhausen integral momentum procedure, while Schlichting (*49*) and Atkinson and Goldstein (*19*) matched a Blasius type of solution with a reverseward perturbation of the fully developed velocity profile. Tomita (*53*) introduced a variational principle to solve the entrance region boundary-layer equation, while Bodoia and Osterle (*5*) used finite difference methods. On the other hand, some authors have avoided boundary-layer methods. Langhaar (*21*) obtained a solution to the capillary tube problem by linearizing the Navier-Stokes equation and by presuming a Bessel function solution. Wang and Longwell (*55*) carried out a full numerical solution of the complete two-dimensional Navier-Stokes equation for several special cases chosen to ascertain the limitations introduced by boundary-layer approximations.

A boundary-layer theory for *purely viscous power law fluids* was developed by a number of researchers (*1, 6, 50*) in the years 1958 to 1960. The classes of possible solutions for flow about submerged bodies were indicated by Schowalter (*50*) and detailed solutions were given by Acrivos, Shah, and Peterson (*1*) and Bizzell and Slattery (*4*). Bogue (*6*) and Kapur and Gupta (*20*) used the integral momentum methods for power law fluids in capillary and flat channel geometries, respectively. Collins and Schowalter extended the Schlichting procedure to analyze the velocity field in the entrance length of a parallel-plate channel (*14*) and the Atkinson-Goldstein method for the cylindrical geometry (*13*), for which case a numerical solution also appears to be available (*10*). Tomita (*53*) used a variational principle for a power law fluid in the entrance region of a tube. [Collins and Schowalter (*13*) pointed out and corrected a numerical error in Tomita's solution.] For reasons noted elsewhere (*26*) the several available studies using the constitutive equation of a Reiner-Rivlin fluid are not of interest.

With one important exception (to be discussed later) these analyses serve nicely to extend some of the body of knowledge developed for Newtonians to simple non-Newtonian fluids. As might be expected, a smooth family of similar solutions is obtained for the various kinds of purely viscous fluid behavior with Newtonian fluids taking the position of one special case in this more general spectrum of fluid behavior.

Of far greater practical importance is the behavior of *viscoelastic fluids* for which the above solutions would not be expected to be applicable, in general. The theory of viscoelastic fluid behavior is almost infinitely more complex,* and while exact solutions are possible in the special case of steady laminar shearing flows, more usually major approximations must be made. If one limits the analysis to flows in which the rates of the changes in deformation rate are small as compared to the relaxation time of the fluid involved (*27*), a series of approximations known as the second, third, etc., order Rivlin-Ericksen type of fluids (*43*) may be derived (*11, 56*). Unfortunately, even these approximations become exorbitantly complex, allowing only perturbation solutions (*9, 22*). An attempt to approximate equations of higher order has been made recently by White and Metzner (*59*) and analyses of the limitations of usefulness of such Rivlin-Ericksen fluids of specific order are given in recent papers by Pipkin and associates (*34, 37*) and by Etter and Schowalter (*15*). While these several studies reveal the difficulties involved in analyzing boundary-layer and entrance length problems for viscoelastic fluids, three noteworthy attempts to do so have been published. Rajeswari and Rathna (*38*) and Beard and Walters (*3*) derived boundary-layer equations for essentially the second-order fluid and investigated the particular case of stagnation point flow. White and Metzner (*59*), using a more general constitutive expression for stress, pointed out the

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* In the remainder of this paper it will be assumed that the reader is familiar with the constitutive and hydrodynamic theories of viscoelastic flow. A well-written history of the development of constitutive theories is given by Rivlin (*41*), while a concise but cogent discussion of this area for chemical engineers was recently published by Schowalter (*51*). A recent book by Fredrickson (*16*) also treats aspects of this subject. The approach and notation of this paper resemble those of our earlier work (*56, 59*) to which the reader is referred.

possibilities of different classes of solutions for flow around submerged objects. No attempt to generalize viscoelastic boundary-layer theory to the entrance length problem appears to have been carried out as yet, and it is to this important question that the remainder of this paper will be devoted.

EQUATION OF MOTION AND PRESSURE FIELD

Consider a channel formed by two horizontal semi-infinite parallel plates separated by a distance $2b$, as shown in Figure 1. A Cartesian coordinate system is located with its origin at the upstream end of the lower plate. The flow is in the direction of the x coordinate, the y axis serves to define positions across the flow field, and the z axis extends along the infinite width of the plate. The velocity field is presumed to be similarly two-dimensional and the fluid to adhere to the plates.

Throughout the inlet region, the equations of motion take the general form:

$$\rho\left[u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right]=-\frac{\partial p}{\partial x}+\frac{\partial t_{xx}}{\partial x}+\frac{\partial \tau_{xy}}{\partial y} \qquad (1)$$

$$\rho\left[u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right]=-\frac{\partial p}{\partial y}+\frac{\partial \tau_{xy}}{\partial x}+\frac{\partial t_{yy}}{\partial y} \qquad (2)$$

If it is assumed, following the usual boundary-layer approximations, that in the immediate vicinity of the channel walls the velocities in the x direction are much greater than those in the y direction, and that gradients in the x direction are correspondingly much smaller than those in the y direction, the velocity field is similar in nature to that surrounding a flat plate (*59*). Therefore, for the fluid near the channel walls the equations of motion take the form

$$\rho\left[u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right]=-\frac{\partial p}{\partial x}+\frac{\partial t_{xx}}{\partial x}+\frac{\partial \tau_{xy}}{\partial y} \qquad (3)$$

$$0=-\frac{\partial p}{\partial y}+\frac{\partial t_{yy}}{\partial y} \qquad (4)$$

while, in the region beyond the "boundary layer" such as at $y = b$, one may write

$$\rho U\frac{dU}{dx}=-\frac{dp(b)}{dx}+\frac{dt_{xx}(b)}{dx} \qquad (5)$$

EVALUATION OF THE ENTRANCE LENGTH

Equations of Motion

Considering the case of moderately high Reynolds numbers (polymer solutions), the entrance length and the shape of developing velocity field may be obtained by extending the boundary-layer methods used previously (*59*). Integrating Equation (4) from the wall to any point within the boundary layer and using this result to eliminate $p(x,y)$ from Equation (3), one obtains

$$\rho\left[u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right]=\frac{d\tau_{yy}(x,0)}{dx}+\frac{\partial[t_{xx}-t_{yy}]}{\partial x}+\frac{\partial \tau_{xy}}{\partial y} \qquad (6)$$

At $y > \delta$, Equation (6) may be expressed

$$\rho U\frac{dU}{dx}=\frac{d\tau_{yy}(x,0)}{dx}+\frac{d}{dx}[t_{xx}-t_{yy}]_{core} \qquad (7)$$

Combining, and neglecting the last term of Equation (7)*

$$\rho\left[u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}-U\frac{dU}{dx}\right]=\frac{\partial \tau_{xy}}{\partial y}+\frac{\partial}{\partial x}(t_{xx}-t_{yy}) \qquad (8)$$

Using the usual methods (*48*) the corresponding integral equation is obtained:

$$(\tau_{xy})_w=\rho\frac{d}{dx}\int_0^\delta u(U-u)\,dy+\rho\frac{dU}{dx}\int_0^\delta(U-u)\,dy+\frac{d}{dx}\int_0^\delta(t_{xx}-t_{yy})\,dy \qquad (9)$$

The integral method of boundary-layer theory will be employed, because, on the one hand, it is impossible to apply the methods of Schlichting (*49*) and Collins and Schowalter (*14*) to obtain better approximations, since no similar solutions to Equation (8) exist for the constant or nearly constant free stream velocities of interest here, and, on the other hand, it will be seen that direct numerical solutions may be premature at this point.

Multiplying Equation (9) through by dx, rearranging, and integrating one obtains for the entrance length

$$L_e=\int_0^{L_e}\frac{\rho}{(\tau_{xy})_w}\frac{d}{dx}\left[\int_0^\delta u(U-u)dy\right]dx+\int_0^{L_e}\frac{\rho}{(\tau_{xy})_w}\frac{dU}{dx}\left[\int_0^\delta(U-u)\,dy\right]dx+\int_0^{L_e}\frac{1}{(\tau_{xy})_w}\frac{d}{dx}\left[\int_0^\delta(t_{xx}-t_{yy})\,dy\right]dx \qquad (10)$$

The details of evaluating the integrals of Equation (10) are similar to the work of Schiller (*47*), Bogue (*6*), and Kapur and Gupta (*20*), especially the latter. Presuming that

$$\frac{u}{U}=f'\left(\frac{y}{\delta}\right)=f'(\kappa) \quad y<\delta \qquad (11a)$$

$$\frac{u}{U}=1 \quad y>\delta \qquad (11b)$$

and defining the integrals

$$\delta^*=\delta_1^*\,\delta=\left[\int_0^1(1-f')\,d\kappa\right]\delta \qquad (12)$$

$$\theta=\theta_1\,\delta=\left[\int_0^1 f'\,(1-f')\,d\kappa\right]\delta \qquad 13)$$

Then observing that by the principle of continuity

$$\int_0^\delta u\,dy+U(b-\delta)=Vb \qquad (14)$$

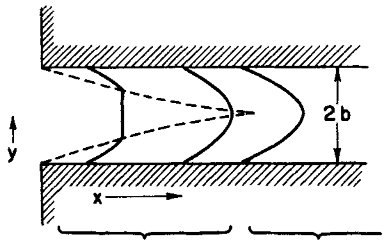

Fig. 1. Development of velocity profiles in entrance region of duct bounded by parallel flat plates.

* Neglect of the last term assumes the "fluid stretching" in the core region to be negligible. This is reasonable for rapid flows in which the deformations in the boundary layer are much greater. Conversely, this would not be a valid assumption for creeping flows, as involved in molten polymers, for example.

one obtains

$$\frac{U}{V} = \frac{1}{1 - \delta_1^* \Delta} \qquad (15)$$

and

$$d\left(\frac{U}{V}\right)^2 = 2\,\delta_1^* \left(\frac{U}{V}\right)^3 d\Delta \qquad (16)$$

Introduction of these results into Equation (10) yields

$$\frac{L_e}{b} = \rho V^2 \int_0^1 \frac{\theta_1 + (\theta_1 + \delta_1^*)\delta_1^*\Delta}{(\tau_{xy})_w (1 - \delta_1^*\Delta)^3}\, d\Delta + \int_0^1 \frac{1}{(\tau_{xy})_w}\, d\left[\Delta \int_0^1 (t_{xx} - t_{yy})\, d\kappa\right] \qquad (17)$$

Constitutive Relationships for Evaluation of Stresses

To proceed further with the evaluation of the integrals of Equation (17), constitutive relationships must be introduced to express the components τ_{xy} and $(t_{xx} - t_{yy})$ of the stress tensor in terms of the characteristics of the velocity field.

In the case of the shearing stress τ_{xy} only the wall value ($y = 0$) is required. In the region close to the wall the velocity field is given by

$$u = U\left[f''(0)\left(\frac{y}{\delta}\right) + \frac{f'''(0)}{2}\left(\frac{y}{\delta}\right)^2 + \frac{f''''(0)}{6}\left(\frac{y}{\delta}\right)^3 + \ldots\right] \qquad (18)$$

Taking the case of an isotropic viscoelastic fluid, which in sufficiently smooth motions may be approximated by a Rivlin-Ericksen type of expansion (*56*), one can determine the shearing stress at the surface by the value of this component of the expansion at $y = 0$. From Equation (18), one sees that at this position

$$u = 0, v = 0 \qquad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 0$$
$$\frac{\partial u}{\partial y} = f''(0)\frac{U}{\delta}, \qquad \frac{\partial v}{\partial x} = 0 \qquad (19)$$

It should be evident that one may show from Equation (19) that

$$(\tau_{xy})_w = \mu f''(0)\frac{U}{\delta} \qquad (20)$$

in which μ denotes the variable laminar shear viscosity of the fluid. For polymeric solutions this quantity is well-defined (*7, 8, 35, 39*) and may be approximated well by any of a variety of simple empirical expressions over a limited range of shear rates. Choosing the power law, one may write:*

$$(\tau_{xy})_w = K\,[f''(0)]^n \left(\frac{U}{\delta}\right)^n \qquad (21)$$

Turning now to the normal stress field one may note that the problem is much more complex, since this information is required over the entire boundary layer, not only at the wall. Normal stresses that have their origin in the viscoelastic properties of the fluid are well known to arise from the first and higher acceleration† tensors and from higher order multiples of the rate of deformation matrix. Acceleration tensors of an order greater than two are identically zero in laminar shear flows. In a boundary layer in which the velocity varies with axial position there is acceleration in the Lagrangian sense and these matrices are not identically equal to zero. From the kinematic analysis of White and Metzner (*59*) and Equation (18), one may, however, show that

$$B_{xx}^{(1)} \sim 0 \quad B_{xx}^{(2)} \sim -2\,[f''(0)]^2 \left(\frac{U}{\delta}\right)^2 \qquad (22)$$

The third acceleration tensor is given by

$$B_{xx}^{(3)} = -u\frac{\partial B_{xx}^{(2)}}{\partial x} - v\frac{\partial B_{xx}^{(2)}}{\partial y} + 2\frac{\partial u}{\partial x} B_{xx}^{(2)} + 2\frac{\partial u}{\partial y} B_{xy}^{(2)} \qquad (23)$$

It is shown elsewhere (*59*) that all these terms are of the same order of magnitude. In the fourth-order approximation, contributions from both $\mathbf{B}_4$ and the product $\mathbf{B}_2^2$ are introduced. From these considerations it follows that in general

$$t_{xx} - t_{yy} = [\alpha + \epsilon][f''(0)]^2 \left(\frac{U}{\delta}\right)^2 \qquad (24)$$

in which α represents the contributions to the normal stress terms arising from the first two acceleration tensors and ϵ is introduced to represent the contributions from acceleration tensors of order higher than two and their products. The evaluation of the terms in ϵ represents a major obstacle, one which was avoided in the three previous analyses of viscoelastic boundary layers (*3, 38, 59*). In the first two cases listed this was done by directly assuming very simple constitutive expressions which contained no acceleration matrices of order higher than two. White and Metzner (*59*) used somewhat more realistic constitutive relationships and proceeded by perturbing about a state of purely viscous non-Newtonian shear flow and, in the degree of approximation used, the objectionable higher order terms also disappeared.

Since no experimental information on the terms involved in ϵ appears to be available, it will have to be neglected in the present analysis. The point of developing the present equations to the degree of completeness indicated by Equations (22) to (24) before doing so was not only to give an appreciation for the consequences of the present approach but, more importantly, to indicate those areas in which further work is required for any improvement in the approximations employed at present.

Discarding the terms in ϵ enables one to turn to data obtained employing laminar shearing flow fields to evaluate the term α. Normal stress-shear rate results that were obtained through 1961 have been reviewed elsewhere (*58*); the several more recent studies (*17, 18, 23, 29, 52, 54*) extend and substantiate the earlier conclusion that usually the normal stresses are proportional to the second power of the shear rate at low shear rates and that the dependence gradually decreases, usually to the first power but sometimes even more greatly at higher shear rates. [This conclusion was first given by Brodnyan, Gaskins, and Philippoff (*7*) on the basis of somewhat scattered but extensive and obviously definitive data.] This behavior, over limited ranges of shear rate, may also be approximated adequately by means of simple empirical equations. By choosing the power law again one gets

$$\alpha = m\,[f''(\kappa)]^{s-2} \left(\frac{U}{\delta}\right)^{s-2} \qquad (25)$$

in which $s \leqq 2$.

These considerations enable one to rewrite Equation (17) as

* Valid objections to use of the power law in any boundary-layer analysis have been noted elsewhere (*26*) It will suffice, however, for the purposes of the present analysis, which is not so much for the purpose of developing precise design information as for giving direction to continued studies of this problem.

† Acceleration tensors were introduced by Rivlin and Ericksen (*43*). The B*n* tensors used in this paper are developed in detail in our earlier papers (*56, 59*).

$$\frac{L_e}{b} = \left\{ 3^{n-1} \left[\frac{2n+1}{3n} \right]^n N_{Re}'' \right\} \int_0^{L_e} \frac{[\Delta^n \theta_1 + (\theta_1 + \delta_1^*)\delta_1^* \Delta^{n+1}] d\Delta}{[f''(0)]^n (1 - \delta_1^* \Delta)^{3-n}} + \left(\frac{2n+1}{n} \right)^{n-s} N_{Ws} \int_0^{L_e} \left[\frac{(1-\delta_1^*\Delta)^n \Delta^n}{[f''(0)]^n} \right] d \left[\frac{\Delta^{1-s}}{(1-\delta_1^*\Delta)^s} \int_0^1 [f''(\kappa)^s dx] \right] \quad (26)$$

in which the Reynolds and Weissenberg numbers (*56, 57, 59*), representing ratios of the inertial to viscous forces and of the elastic to viscous forces, respectively, have been introduced.

Velocity Profiles

The remaining problem is to identify a suitable velocity field Equation (11). This is subject to the restrictions that

$$f'(0) = 0 \qquad f'(1) = 1 \qquad f''(1) = 0 \qquad (27\ a, b, c)$$

and that the fully developed profiles at $x = L_e$ coincides with the usual power law distribution. With the assumption of a cubic polynomial, Equations (27) lead to

$$f' = a\kappa + (3 - 2a)\kappa^2 + (a - 2)\kappa^3 \qquad (28)$$

in which the parameter a remains to be defined. Following Bogue (*6*) and Kapur and Gupta (*20*) one may note that taking

$$a = \frac{n+1}{n} \qquad (29)$$

enables one to fit the fully developed distribution rather well for $n > 0.5$. This satisfies the requirement that the fully developed distribution must be independent of the elastic properties of the fluid, but implies that the developing profile should be independent of elasticity also. Other work (*59*) indicates this cannot be strictly the case but it will be used here as an initial approximation.

Entrance Length Magnitude

Combination of Equations (26), (28), and (29) gives

$$\frac{L_e}{b} = N_{Re}'' \left[\frac{3^{n-1}}{420\, a^n} \left(\frac{2n+1}{3n} \right)^n \left(\frac{12}{6-a} \right)^{n+1} \right] [(318 - 17a - 8a^2)\, B\delta_1^*\,(n+2, n-2) + (54 + 9a - 4a^2)\, B\delta_1^*\,(n+1, n-1)] + N_{Ws}\, \psi \left(\frac{2n+1}{n} \right)^{n-s} \frac{1}{a^n} \left(\frac{12}{6-a} \right)^{n+1-s} [(1-s)\, B\delta_1^*\,(n+1-s, n+1-s) + s\, B\delta_1^*\,(2+n-s, n-s)] \quad (30)$$

in which the $B\delta_1^*$ terms denote incomplete beta functions defined by

$$B\delta_1^*\,(p, q) = \int_0^{\delta_1^*} x^{p-1}\,(1-x)^{q-1}\,dx \qquad (31)$$

Equation (30) may also be rewritten as

$$\frac{L_e}{b} = \text{٣}(n)\, N_{Re}'' + \text{٣}(n,s) N_{Ws} = \left(\frac{L_e}{b} \right)_{pv} + \text{٢}(n,s) N_{Ws} \qquad (32)$$

Herein the first term represents the entrance length for purely viscous fluids. This term is identical to that of Kapur and Gupta† (*20*). Correspondingly, the term containing the Weissenberg number represents the incremental change in the entry length due to elastic effects; Equation (30) shows that the coefficient ٣ (n,s) may be positive, negative, or zero, depending upon the magnitude of the parameters s and n.

Typical values of the coefficients ٢ and ٣ are given in Table 1. The particular combinations of n and s which have been chosen are those which have been found to portray the properties of various real fluids (*18, 52*) over some variety of deformation rates. The negative values of ٣ must be used with judgement so that one can remain within the domain in which the boundary-layer assumptions employed continue to be valid, that is, to satisfy the inequality $L_e/b >> 1$.

Turning to a comparison of the magnitudes listed in Table 1 with experimental values, only qualitative observations are possible, since, in many cases, no fluid property data from which the magnitudes of the Weissenberg number may be calculated are available and in all cases the available data are for flow through round tubes, not for parallel plate channels. However, the latter difference should not void qualitative conclusions, and sufficient normal stress measurements are available to assess the general magnitude of the Weissenberg number; for highly viscoelastic polymer solutions possessing low values of the parameters n and s it is likely to be in the range of 10 to 100 under conditions of interest (*7, 17, 18, 23, 28, 52, 54*). For such fluids, reference to Table 1 shows that appreciable increases in length of the entry region are predicted. A greater entry length for such viscoelastic fluids is a well-defined experimental fact (*25, 30, 36, 44, 45*)‡ but the experimental magnitudes are very appreciably greater than those calculated. It should be noted that the power law approximations embodied in Equations (21) and (25) are certainly not likely to be in error by this magnitude, nor, in general, even in this direction. Similarly, while integral momentum methods apparently may not be as accurate for non-Newtonians as for Newtonian fluids (compare *6* and *13*), they have been found generally to enable useful predictions and it is not clear that the same could not have been expected here. The difficulty would therefore appear to arise from either the use of the boundary-layer approximations [Equations (3) and (4)] or due to the general choice of the constitutive relations used to depict fluid properties.

Considering first the chosen constitutive relationships, two general questions must be considered. In the first instance, the contributions of the higher order acceleration tensors [the terms designated by the ϵ of Equation (24)] were neglected, since no information is available concerning these. While this may be a source of some of the difficulty, inclusion of terms of progressively higher order extends the range of the simpler equations only infinitesimally (*17*) and obviously this is not a useful direction in

† This term has been evaluated more carefully by Collins and Schowalter (*14*) than is possible when integral methods are used and, for design, purposes, their value might well be used in preference to the formulation given in Equation (30).

‡ Very great entry lengths have also been observed in molten polymers (*2, 31*) but these occur at very low Reynolds numbers beyond the expected region of validity of the boundary-layer approximations used here.

TABLE 1. COEFFICIENTS OF THE REYNOLDS AND WEISSENBERG NUMBERS

n	s	٢(n)*	٣'(n,s)
0.5	0.5	0.122	0.288
0.5	1.0	0.122	0.062
1.0	1.0	0.138	0.108
1.0	1.5	0.138	−0.193
1.0	2.0	0.138	Analysis breaks down

* As given in reference *14*.

which to proceed even if this turns out to be the problem. Second, the general Rivlin-Ericksen expansions are valid only for sufficiently smooth flows—flows for which the characteristic time of the flow field is appreciably greater than that of the fluid. This is considered in more detail elsewhere (*27*) and it may be shown that this condition is not met just at the entry of the channel.

The type of material behavior which must be considered at the entry to the channel may be developed by means of several alternative approaches. First, using a constitutive equation of the form (*58*)

$$\tau'^{ij} + \theta \frac{\delta \tau'^{ij}}{\delta t} = \mu d^{ij} \tag{33}$$

in which $\theta = \mu/G$ and $\tau^{ij} = -pg^{ij} + \tau'^{ij}$, and which appears to portray rapid laminar shearing flows and some stress relaxation phenomena moderately well (*27, 58*), one may note that in the case of flows for which the time function θ is much greater than the time constant of the flow field, the convected derivative term is much greater than the first stress term, which may therefore be dropped as an approximation. This leads to

$$\tau^{ij} = -pg^{ij} + Ge^{-\Delta t/\theta}\,\epsilon^{ij} \tag{33a}$$

in which ϵ^{ij} denotes the Piola-Finger strain tensor (*56*). This particular strain tensor arises because of the use of the contravariant deformation rate tensor in Equation (33). Thus the stress is given by an equation having more nearly the general form of those describing the behavior of elastic solids than of those commonly associated with fluids.

An alternate formulation may also be given. For the case in which the stresses are due to smoothly increasing deformations over a time interval Δt which is smaller than the relaxation time, it may be shown (*60*) that it is possible to approximate these by means of relations of the kind

$$\tau = -p\mathbf{I} + \sum_n \frac{(-1)^{n+1}}{(n-1)!}\left[\int_0^{\Delta t} s^{n-1}\, G(s)\, ds\right]\mathbf{B}_n + \ldots. \tag{34}$$

In fluid fields for which

$$u = \Gamma y$$
$$v = w = 0 \tag{35a, b}$$

this leads to

$$\tau_{xy} = [G(0) - a_1\Delta t + \ldots](\Gamma \Delta t) + \ldots. \tag{36a}$$

$$\tau_{xx} - \tau_{yy} = \left[G(0) - \frac{a_1}{2}\Delta t + \ldots\right](\Gamma \Delta t)^2 + \ldots. \tag{36b}$$

when the relaxation function is approximated by an equation of the form

$$G(\Delta t) = G(0) - \sum_{a=1} a_a(\Delta t)^a \tag{37}$$

As the term $\Gamma\Delta t$ in Equation (36) is again simply the shearing strain, the stresses here are also given by relations equivalent to those of nonlinear elasticity (*16*).

The above arguments show that just at the entry to the channel a viscoelastic fluid behaves as a solidlike material. Thus, the usual portrayal of a two-part developing flow field (Figure 1) should perhaps be replaced by one consisting of three distinct regions as indicated in Figure 2; that is, a nearly solidlike region near the entry followed by a developing boundary layer further downstream, and, finally, by the region of well-developed flow. In the case of Newtonian fluids the relaxation times are so small that the first of these three regions perhaps may be neglected as a not unreasonable approximation (*55*). On the other hand, for polymeric materials the relaxation times are so great (*27*) that this solidlike* region may be of primary importance and its omission in the preceding analysis may well be the source of the discrepancy between calculation and experiment. One may note that while the basic field equations [Equations (1) and (2)] would be expected to apply within such a semirigid region, the simplifying assumptions used to obtain the boundary-layer equations [Equations (3) and (4)] and the boundary conditions employed with Equation (9) would both be invalid.

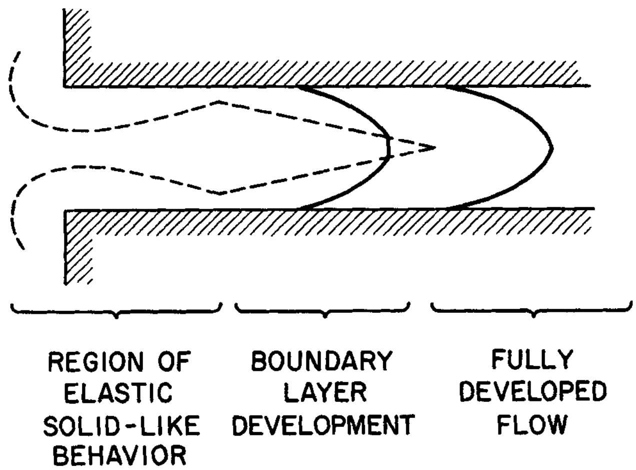

Fig. 2. Development of velocity profiles in highly viscoelastic fluids (or in highly dilatant fluids).

It should be noted that under extreme conditions the solidlike regions may extend across the entire duct, thus closing the entry and requiring either fluid fracture or slip of the fluid at the tube walls for continued flow. The relationship of these considerations to a mechanistic interpretation of the "melt-fracture" phenomenon is clear. Finally, it may be noted that similar solidlike regions, leading to abnormally high entrance pressure losses and ultimately to fluid fracture, would also be expected in the case of purely viscous fluids which are highly dilatant in the rheological sense. In this event the viscosity coefficient may increase so rapidly with increasing deformation rate as to render the material effectively solid and therefore unable to accommodate the high deformation rates at the entry implied by Figure 1. Such abnormal pressure loss effects and fracture phenomena have indeed been observed in dilatant slurries (*32, 40*) and their similarity to the behavior of viscoelastic materials has been noted (*33*). However, the reasons for the similar behavior of such radically different kinds of fluids has remained unexplained.

Thus, these observations provide, apparently for the first time, a firm and detailed description capable of mathematical representation of the flow phenomena responsible for fluid fracture and related instabilities in a variety of fluid materials. They also serve to define clearly the desirable directions for further analyses in this area. Finally, they indicate that previous boundary-layer analyses for purely viscous fluids may be satisfactory for materials having low flow behavior indices but that these become increasingly subject to error as the flow behavior index increases appreciably above unity.

CONCLUDING REMARKS

Two primary conclusions appear to be possible. First, viscoelastic and dilatant fluids may behave like quasi-

* Note added in proof: To avoid possible misinterpretation, it should be emphasized that the term *solidlike* was not intended to imply a perfectly rigid solid, but rather a region of significantly decreased fluid deformation rates in which the material properties take on *some* of the characteristics more usually attributed to solids.

elastic solids in the immediate vicinity of the duct entrance and at the leading edge of any blunt object. This fact, with its implications of a finite "boundary-layer" thickness at the entrance or leading edge, as depicted by Figure 2, will have to be considered carefully in further analyses. Evidently it is the neglect of this behavior which leads to the discrepancy between the calculated value of the coefficient Υ (n,s) of Table 1 and the values noted experimentally. Second, beyond the immediate vicinity of the entrance, the fluid behavior should gradually become describable by the Rivlin-Ericksen type of constitutive relations. The present analysis indicates that the solution in this region of the duct will be rather sensitive to the detailed fluid properties (n and s of the present analysis); thus, while the general Rivlin-Ericksen equations may portray adequately the fluid properties in this region, simple approximations such as those of the second or third order that were used frequently in the past are not likely to be of value.

ACKNOWLEDGMENT

This work has been supported by the Office of Naval Research and its reproduction, in whole or in part, is permitted for any purpose of the United States Government.

NOTATION

a = velocity profile parameter, Equation (29)
a_a = coefficients in series expansion, Equations (36) and (37)
b = centerline-to-surface distance, that is, spacing of plates = 2b
$B\delta_1^*$ = incomplete beta function, Equation (31)
$B_{(n)}$ = n^{th} order acceleration tensor; defined in detail in references 56 and 59
$B_{xx}^{(n)}$ = the xx component of B_n
f' = dimensionless velocity, $u/U \cdot f^{(n)}$ denotes the $(n-1)$ derivative of f' w.r.t. κ
$G(s)$ = relaxation modulus of linear viscoelasticity
K = consistency index, Equation (21)
L_e = entry length (region extending from duct entry to onset of well-developed flow in Figures 1 and 2)
m = elasticity index, Equation (25)
n = flow behavior index, Equation (21)
N_{Re}'' = generalized Reynolds number for parallel plate geometry. $N_{Re}'' = \dfrac{b^n V^{2-n}\rho}{K\left(\dfrac{2n+1}{3n}\right)^n 3^{n-1}}$ for power law fluids (see reference 57)
N_{Ws} = Weissenberg number, $N_{Ws} = \dfrac{m}{K}\left(\dfrac{3V}{b}\right)^{s-n}$
p = hydrostatic or isotropic pressure
s = elasticity characterization index, Equation (25)
t_{xx}, t_{yy} = deviatoric normal stresses in the x and y coordinate directions, respectively
u,v = local velocities in the x and y coordinate directions, respectively
U = centerline velocity
V = bulk (average) velocity
x,y,z = coordinate labels
$\Upsilon(n)$ = coefficient of Reynolds number term, Equation (32)
$\Upsilon(n,s)$ = coefficient of Weissenberg number term, Equation (32)
α = material property coefficient arising from first two acceleration tensors [Equation (24)] approximated by Equation (25) later
Γ = shear rate
δ = boundary-layer thickness
δ_1^* = integral defined by Equation (12). The displacement thickness of the boundary layer is equal to $\delta_1^*\delta$
Δ = dimensionless boundary-layer thickness, δ/b
ϵ = contribution to normal stresses arising from the higher order terms in velocity
κ = dimensionless y coordinate, y/δ
θ = fluid relaxation time function, Equation (33)
θ_1 = integral defined by Equation (13). The momentum thickness of the boundary layer is equal to $\theta_1\delta$
μ = fluid viscosity (variable in the non-Newtonian case)
ρ = fluid density
τ_{xy} = shearing stress
ψ = dimensionless constant, $\psi_1 = \int_0^1 [f''(\kappa)]^s \, d\kappa$

Subscripts

pv = purely viscous
w = wall conditions

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Manuscript received January 25, 1965; revision received June 8, 1965; paper accepted June 8, 1965.

Entrance Region Flow

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The results of rigorous numerical solutions of the general equations of motion are presented for isothermal, laminar, Newtonian flow in a tube entrance region for a uniform entrance velocity. Computed velocity profiles, entrance lengths, and pressure gradients are compared with previous theoretical and experimental results.

In flow from a chamber up to and through a tube, the fluid velocity profile, which at the entrance may vary considerably with conditions, approaches asymptotically that of fully developed flow in the entrance region. The pressure field in a tube entrance region differs from that for fully developed flow in consequence of the kinetic energy changes and the excess viscous dissipation of energy which occur in the establishment of the developed velocity profile. An exact description of the velocity and pressure fields in the entrance region would provide a basis for evaluation of the many approximate solutions of the equations of motion and boundary-layer equations for entrance flow, for analysis and understanding of tube entrance region forced convection energy and mass transport, and for development and evaluation of flow stability theory.

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* In accord with usual practice the entrance length or region is here defined to extend from the beginning of the constant diameter section of the tube to a point where the centerline velocity has increased to 99% of that for fully developed flow.

This study is concerned with Newtonian flow in the entrance region of a tube of circular cross section in which the velocity profile at the tube entrance is approximately flat, such as may occur in viscous flow into a tube with a rounded entrance at relatively high N_{Re}.

In general, solutions of the differential equations of motion for tube entrance flow consist of approximate solutions of restricted forms of the equations of motion, variations in the application of the Prandtl boundary-layer equations, or combinations of these in which a boundary-layer solution valid near the entrance is coupled with a solution of restricted equations of motion which is valid far from the entrance.

By restricting application of the equations of motion in cylindrical coordinates (*1*, p. 85) to conditions such that the flow is independent of time, the radial component of the equations of motion is negligible, any angular motion is negligible (axisymmetric flow), the fluid density and viscosity are constant (isothermal and isobaric flow or temperature and pressure independence), and the flow is